Contracting Except'l Divisors ctd

<u>MMP & KRF</u>: (X, ω_0) = projective manifold

$g_{i\bar{j}} = -R_{i\bar{j}}$, $\omega = \frac{i}{2\pi} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$, $[\omega] = c_1(L)$ ample.

$\dot{\omega} = -Ric \quad \Rightarrow \quad [\omega(t)] = [\omega_0] + t\, c_1(K_X)$

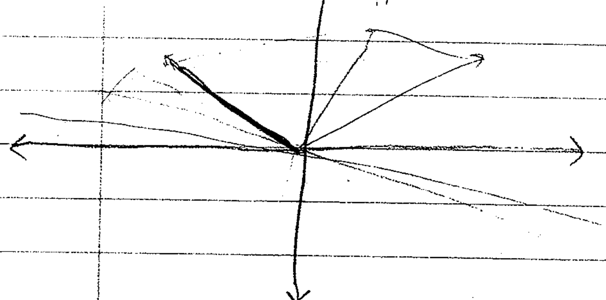
(<u>Thm</u> (C_∞): Flow exists as long as $[\omega_0] + t\, c_1(K_X) > 0$)

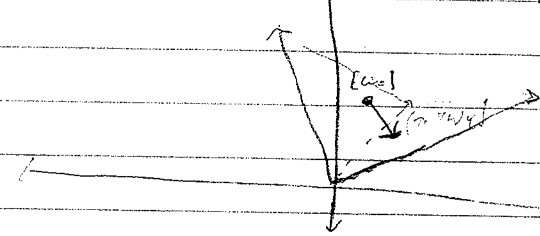
Say $T < \infty$ s.t. $[\omega_0] + T c_1(K_X)$ nef, not positive.
$\Rightarrow \exists$ curve $C \subset X$ s.t. $\int_C \omega(T) = [\omega(T)] . C = 0$
Kawamata: $T \in \mathbb{Q}$, so $[\omega_0] + T c_1(K_X) = [L + T K_X] \in \mathbb{Q}$-div
<u>and</u> semi-ample (b.p.f.)

$\Rightarrow |r(L + T K_X)|$ gives map $\pi_C : X \to \mathbb{P}^n$,

$(L + T_0 K_X) . C = 0 \Rightarrow$ map contracts C, at (very) least.
(map doesn't see C.) ~~(at very least.)~~

$\overline{NE}(X)$ (eff. curve cone.) $\qquad NEF(X) \subset [X, \mathbb{R}]$ (divisors/num eq.)





Planes $[\omega(t)]^\perp$ move until hit extremal ray.
(<u>Will happen</u> if K_X not nef: def'n of extremal rays.)
(in fact $\perp$ to whole wall.)

If starting class was generic, hit a wall
of max'l dimension in $NEF(X) \Rightarrow$ contract
exactly one ray.

Do the proj'n, restart flow. [Dimension of some collapses, C goes away, (back in Kähler cone) your V positive again. Repeat. ENDGAME: Find KE metric or ... is (def'n.)]

②

Problems: Good theorem, dim > 3?

~~Algebra~~ · What's π_c? — Contraction, flip, fibering... dim > 3? ~~flipme~~ / red. divisor (SAY DEF)

· $\pi_c(X)$ often singular, what's a metric? What's Ricci flow? (Weak KRF)

— Tian: do degenerate RF on $\tilde{X}$... but that's not a step.. (SW II: orbifolds distinct from $\tilde{X}$)

- What's K_X? · Does semi-ampleness hold for X singular? Rationality?

— Can do one blow-down to singular, but next step?

Kodaira's Lemma? · Analysis ✓ limiting metric Descends to LP, nice enough to restart flow? b/c right cohomology — Condition A?

Need to do rigor first? · Geometry: get GH convergence (lacking metric completion $(X) = Y$) but not yet for flips. Currently not even topology works

(· "Analytic surgery" = do-it-yourself.)
(· "Canonical surgery"

Def: ~~Canonical Surgical Contraction~~

Note: GH? 18 cases to replace GH blowup! Kodaira's lemma is just finite order! Not need

~~Defining Subset for "Effective" divisor, nec. point.~~

(Let's just do analysis and see what we can recover of Y.)

Let $\pi: X \to Y$ contract $E = \mathbb{P}^1$, ω_0 Kähler s.t. $[\omega_0] + T c_1(K_X) = [\pi^* \omega_Y]$

and take $\hat{\omega}_t = \frac{(T-t)\omega_0 + t\pi^*\omega_Y}{T}$, $\partial \hat{\omega}_t = \ldots$

Let $\omega(t) = \hat{\omega}_t + i\partial\bar{\partial}\varphi$, $\omega(0) = \omega_0$ be RF

$\Rightarrow \dot{\varphi} = \log(\hat{\omega}_t + i\partial\bar\partial\varphi)^n / \Omega$

To get uniform estimates for ... (Based notes will be ...)

Uniform estimates for $t \in [0, T)$: (Tian-Zhang) ③

2.1) $\|\varphi\|_{C^0} \le C$

$\dot{\varphi} \le C$

$\omega^n \le C\,\Omega$ — fixed vol. form on X

$\varphi_{(t)} \to \varphi_T$ pointwise, bdd $\pi^*\omega_Y$ p.s.h. → closed pos. (1,1)-current.

(proof: max princ, etc)

2.2) (i) $\omega \ge c\,\pi^*\omega_Y$ (best one can bound e-val's from below.)

(ii–iv) For $K \subset X \setminus E$ cpct, ω C^∞ bdd & converges in C^∞ $\boxed{\text{on } K}$ to smth Kahler ω_T.

f hol'c on π, $f'(0) \ne 0$, $|K'|\le |f'(0)|$. ... Thm ... using

Proof: (i) (Use the) Schwartz, (ii–iv) $\pi^*\omega_Y \sim \omega_0$ on K, so (i) ⇒ e-val's bdd below (& above) std parabolic theory.

$(\partial_t - \Delta)\log \operatorname{tr}_{\omega}\omega \le C\operatorname{tr}_{\omega}\omega_0$

Cor: ω satisfies "Condition A" of S-T (they failed to ... (effective, union prime, defining section: ... (diff Schwartz?))

Pf: $\omega \overset{(i)}{\ge} \pi^*\omega_Y \ge |s_E|^{2k}\omega_0 = |s_{k\cdot E}|^2\omega_0$; some k,

just b/c π algebraic ⇒ gives finite power (conf.)

(I wondered for awhile just whether this holds, whether the inv. image is reduced.) (Later in S-T, see $k = 1$ in our case: Kodaira CGM)

Rmk: This all works for $\pi : X \dashrightarrow Y$ ... & $[\omega(T)] = [\pi^*\omega_Y]$

smooth, $\mathbb{Q}$-fact. terminal sing's?

Paper II: "smooth" orbifolds.

(RF limit s.t. coh. descends to an ample class)

This is already enough to continue the flow using weakly S-T.

(4)

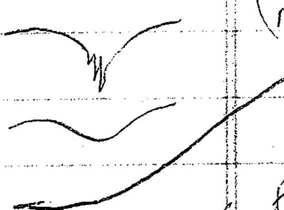
Just remark that since $\hat{\omega}_T|_E = \pi^* \omega_Y|_E = 0$,
$\frac{i}{2\pi}\partial\bar\partial\varphi_T|_E = \omega(T)|_E \geq 0$, so max. princ.
$\Rightarrow \varphi_T = $ const
& $\omega(T)|_E = 0$.

$\Rightarrow \varphi_T$ descends to ψ_T bdd on Y (pluripot'l $\Rightarrow L^1$ theory, Kolodziej cts.)
s.t. $\omega' := \omega_Y + \frac{i}{2\pi}\partial\bar\partial\psi_T \geq 0$
(closed, pos. 1,1 current) (needed for flow conv.)
And $\left(\frac{\omega'}{\omega_Y}\right)^n \in L^p(Y)$, $p>1$, as before b/c $\omega(T) < \Omega$ on X

$$\left(\int_{Y \setminus \{y_0\}} \left(\frac{\omega'^n}{\omega_Y^n}\right)^p \omega_Y^n = \int_{X \setminus E} \left(\frac{\omega(T)^n}{(\pi^*\omega_Y)^n}\right)^p (\pi^*\omega_Y)^n \right.$$

$$\left. \leq C \int_{X\setminus E} \left(\frac{\Omega}{(\pi^*\omega_Y)^n}\right)^{p-1} \Omega < \infty \right)$$

So T: flow continues, smooth & nondeg. for $t > T$.
(SW II: Becomes nondegen. smooth orbifold metric for $t>0$ (better than degen. flow on singular X.))



MA eq'n changes and Thm $\not\Rightarrow$ restarted flow sat's RF eq'n (~~...~~) at $t=T$ (initial)

Need higher-ord. est's (PSS)-type. ~~With any, given time~~
(How generally can they do them? Flips?)

For higher order, need to control deriv's & curv. as $t \to T^-$, $x \to E$, in order to (have a chance to) control the approximating flows ω_ε (on the other side) as $t \to T^+$, to show $\omega'(t)$ solves RF at $t = T$ (just need $\omega'(T^+)$ exists.)
Discuss ~~...~~ $t\to T^-$
Dan-style $t \to T^+$.

Now: Does this do ... MMP? **Def:** Canonical sing. cond.

$E = \cup D_i$ ... and ... $\pi(D_i) = y_i$ (i) smooth ... X ... (ii) Spaces $(Y, d_{T}) \simeq (Y, \omega_{Y})$ (iii) ... (iv) ... (v) ...

Had: crude upper & sufficient lower bd to continue the flow.

(Want real control, metric $\mathcal{D} \to 0$ ... on/near E.)

Recall blowing (setup): charts $\widetilde{D}_i \to D_i$

$E \cap \widetilde{D}_i = \{z(i)^i = z^i = 0\}$ $\quad (z(i)^j) \to z(i)^i(\ldots, z(i)^j, \ldots, 1, \ldots)$

Section s of $[E]$ given by $s|_{\widetilde{D}_i} = z^i$ (part of 1.)

Define metric h s.t. $|s|_h^2\Big|_{D} = \sum |z^i|^2 =: r^2$ 1 on D_i

$R(h) = -\frac{i}{2\pi}\partial\bar\partial \log(\sum |z^i|^2) \Rightarrow \omega_X := \pi^*\omega_Y \overset{D_{1/2}}{=} \varepsilon_0 R(h)$ is Kähler. (Kodaira) $\left(\int_E R(h) = E^2 = -1\right)$

7.5. Key Estimates: (i) $c_0 \pi^* \omega_Y \le \omega(t) \le \frac{C}{|s|_h^2} \pi^* \omega_Y$ $\quad (\omega_X$ ... $=$ ... $)$

(ii) $\frac{|s|_h^2}{C} \omega_0 \le \omega(t) \le \frac{C}{|s|_h^{2(1-\delta)}} \omega_0$

GH convergence. ~~Chegalated KRF ...~~ (using KRF ... ~~rather than~~ ... purpose ~~...~~ structure.)

Lemma 7.7 (state of the art)

(i) $\mathrm{Diam}(S_r) < C$, $0 \le r < \frac{1}{2}$

(ii) $x \in D_{1/2}$, $\gamma(\lambda) = \lambda \cdot x$, $\lambda \in (0, 1]$ $\quad \boxed{\Longrightarrow \mathrm{diam}_g(X) < C}$

$\mathrm{length}_g(\gamma) \le |x|^{1/2} \cdot C$.

Pf. $\omega_Y \sim \omega_{Eucl}$ on D $\Rightarrow \omega \le \frac{C}{r^2}\omega_{Eucl}$ ((i) above.)

(i) $d_g^{S_r}(p, q) \le \sqrt{C}\, r^{-1} d_g^{S_r}(p, q) \le \sqrt{C}\,\pi$.

(ii) $\int_0^1 \sqrt{g(\gamma'(\lambda), \gamma'(\lambda))}\, d\lambda \le C \int_0^1 \left(\frac{|x|}{|\lambda x|^{(1-\delta)}} d\lambda\right) = \frac{C|x|^\delta}{\delta}$ ((ii) above)

If you redo 7.5 with $\omega_Y = \omega_{Eucl}$ on $D_{1/2}$,

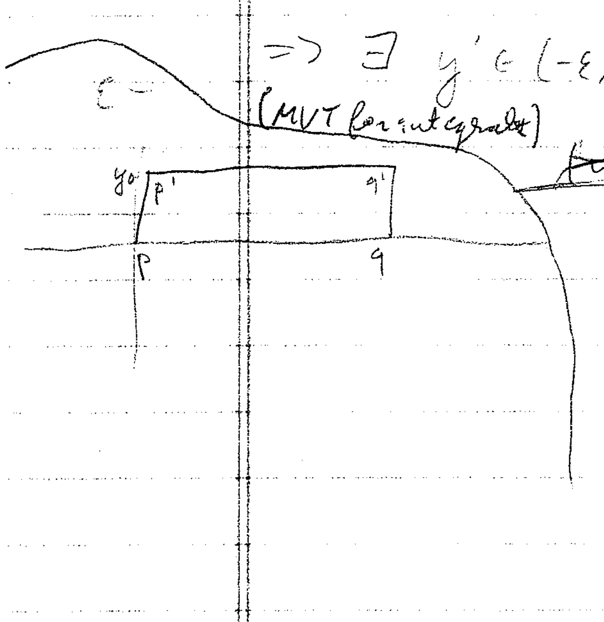
$Rm(\omega_Y) = 0|_{D_{1/2}} \Rightarrow$ get $(\partial_t - \Delta)\log \mathrm{tr}_{\pi^*\omega_Y}\omega \le 0$,

(sectional curv's) (your Schwartz gets even bigger) $\rightsquigarrow \delta = \frac{1}{2}$.

<u>Lemma 3.2</u> $\exists$ uniform C s.t. $\forall\, p,q \in E$, $t \in [0,T)$,
$$d^X_{\omega}(p,q) \le C(T-t)^{1/3}$$

<u>Proof</u> Assume $E = \mathbb{P}^1$, $\dim X = 2$.
$$\int_E \omega(t) = \frac{1}{T}\int_E \left((T-t)\omega_0 + t\,\pi^*\omega_Y\right) = \frac{T-t}{T}\int_E \omega_0 \le C(T-t).$$

Let $\varepsilon = (T-t)^{1/3}$ suff. small. (statement was equiv. to suff. small $(T-t)$.)
Assume w/og $p,q \in B_2(0) \subset \mathbb{C}^2$ and $\overset{\text{spot}}{\text{same}}$ extend.
b/c in stereo. coords on $\mathbb{P}^1 = S^2$, this is bigger than a hemisphere. By compactness,
ω_{Eucl} on $B_2(0)$ is unif. equivalent to restriction of g_0 to B_2 after any rotation (putting $p,q \in B_2(0)$).
Moreover assume $p = 0$, $q = (x_0, 0)$, $0 < x_0 < 1$,
and $R = \{0 \le x \le x_0,\ -\varepsilon \le y \le \varepsilon\} \subset B$
$$\int_R \mathrm{tr}_{g_0} g\, dx\, dy = \int_{-\varepsilon}^{\varepsilon}\int_0 g_0^{-1} g\, dx\, dy \le C\int_E \omega \le C(T-t)$$
$(g_0, g \in R)$
$$\Rightarrow \exists\, y' \in (-\varepsilon,\varepsilon) \text{ s.t. } \int_0^{x_0} \mathrm{tr}_{g_0} g(x,y')\,dx \le \frac{C}{\varepsilon}(T-t) = C(T-t)^{2/3}$$
(MVT for integrals)



~~(wt ...)~~

$$d_\omega(p',q') = \int_0^{x_0} \sqrt{g(\partial_x,\partial_x)}\,(x,y')\,dx$$
$$= \int \sqrt{\mathrm{tr}_{g_0} g}\,\sqrt{g_0(\partial_x,\partial_x)}\,(x,y')\,dx$$
$$\le \left(\int_0^{x_0} \mathrm{tr}_{g_0} g(x,y')\right)^{1/2}\left(\int_0^{x_0} g_0(\partial_x,\partial_x)(x,y')\,dx\right)^{1/2}$$
$$\le C(T-t)^{1/3}$$

For the other 2 sides, have to go outside E.

$$d_\omega(p,p') + d_\omega(q,q') \le C\varepsilon$$

$d_\omega(q,q') \le$ $\qquad + d_\omega(p,q') \le 2C\varepsilon$.

(unif. bdd) $\cdot\,\varepsilon$ by key est. as per L 2.7(i)

sub. small, 2.7(ii)

For $\mathbb{P}^{n-1} \subset X^n$, take a $\mathbb{P}^1$ cont. x & y. ∎

Seems have no control, however, on ω_T at E except via its integral

(... could have geodesic disks whose circles fail to go to 0?)

Certainly could miss the fixed cE distance $\not\to 0$.

Messes up $(X \setminus E)_{g_T} \simeq (Y, d_T)$. Can you have [illegible] $d\theta^2$, so missing the origin matters?

(Extend ω_T by zero at y_0 $=: d_T$.)

For G-H, guess that the maps are? needn't be continuous. "Analytic category?" ~~it but~~ G-H limit is unique.

METRIC COMPLETION ISSUE.

(Right now, appears that KRF is superior [illegible], and v.v.)

→ can KRF be used to organize huge amt of data? Opposite.

(RF was used to ignore structure, but on topology. Still they may interact strongly!)

(Return to p.4 discussion of higher order
= last part of "canonical surgical contraction.")

Still have Smoothness at $t = T$ (away from $E \sim y_0$).

Prop: $S \le \frac{C}{(S)}$ (2A)

Illustrating from PSS C^3 est's (control Γ first.) 8

$h^\alpha_\beta = g_0^{\alpha\bar\gamma} g_{\bar\gamma\beta}$ $\quad B_j{}^\alpha{}_\beta = (\nabla_j h^\alpha{}_\gamma (h^{-1})^\gamma{}_\beta)$ $\quad \boxed{S = |B|^2_g}$ ~ heat eqn

$h^{-1} = g^{\alpha\bar\gamma} g_{0\bar\gamma\beta}$ $\quad \hat R = Rm(g_0)$ $\quad = A_j{}^\alpha{}_\beta - \hat A_j{}^\alpha{}_\beta$ (norm. coords for g)

Also: $\dot h$ $\quad h^{-1}\dot h = -g^{\alpha\bar\gamma} R_{\bar\gamma\beta} = -R^\alpha{}_\beta$

unnorm'd: $\phi_{j\bar k m} = g_{\bar k\alpha}(\nabla_m h h^{-1})_j$

$$\Rightarrow \partial_{\bar k} B_j{}^\alpha{}_\beta = \hat R_{\bar k j}{}^\alpha{}_\beta - R_{\bar k j}{}^\alpha{}_\beta$$

$$\Delta B = \nabla^{\bar p}\partial_{\bar p} B_j{}^\ell{}_m = -\nabla^{\bar p} R_{\bar p j}{}^\ell{}_m + \nabla^{\bar p}\hat R_{\bar p j}{}^\ell{}_m$$
$$= -\nabla_j R^\ell{}_m + \nabla^{\bar p}\hat R_{\bar p j}{}^\ell{}_m$$

no barred indices.

$$\Delta S = (\Delta B, B) + (B, \bar\Delta B) + |\bar\nabla B|^2 + |\nabla B|^2 \qquad \Delta - \bar\Delta.$$
$$= -(\nabla R, B) - (B, \nabla R) + |\bar\nabla B|^2 + |\nabla B|^2 + B * R * B$$
$$+ (\nabla^{\bar p}\hat R_{m\bar p}, B) + (B, \nabla^{\bar p}\hat R_{m\bar p} -)$$

raising does 'diff. generous

Since Yau knew Ricci, and $\nabla^{\bar p} = O_g(\nabla h h^{-1}) = O_g(B) = O(S)$,
get $\quad \Delta S \ge -C_1 S - C_2$ for Calabi-Yau eqn.

(throw in ϕ to flip sign, C^0, C^2 bdd) ~~(they used a $\hat\Delta\phi$ for ...)~~

We need to cancel ∇R & R with ∂_t, under RF.

Claim: $\dot B_{m}{}^\alpha{}_\beta = \nabla_m (h^{-1}\dot h)^\alpha{}_\beta = -\nabla_m R^\alpha{}_\beta$

Pf:

$$(\nabla_j \dot h)^p{}_q = \{ g^{-1}\partial_j (g \underbrace{h g^{-1}}_{const}) g\}^p{}_q \qquad \dot g = g h^{-1}\dot h, \quad \dot{g^{-1}} = -h^{-1}\dot h g^{-1}$$

$$= -h^{-1}\dot h \nabla_j h + \nabla_j h (h^{-1}\dot h) + \nabla_j (h^{-1}\dot h h) = \nabla_j (h^{-1}\dot h) h + \nabla_j h (h^{-1}\dot h)$$

Now mult. by h^{-1}. Cor: KRF = YM flow. (2nd Bianchi)

use to don't $|Rm(t)|^2 \le T$ $\quad \frac{1}{t} \Delta |R| \le C \ldots$

$$\Rightarrow (\Delta - \partial_t) S = (-\nabla_m R^\alpha{}_\beta + \nabla_m R^\alpha{}_\beta, B) + \cancel{(\quad)} + |\bar\nabla B|^2 + |\nabla B|^2$$
$$+ 2\mathrm{Re}(\nabla^{\bar q}\hat R_{m\bar q} -, B) \qquad \cancel{+ B * (R \ldots + \partial_t g) B}$$

RF miracle.

$$\boxed{(\partial_t - \Delta) S \le -|\bar\nabla B|^2 - |\nabla B|^2 + C_1 S - C|\nabla Rm(g_0)|^2}$$

I'd said, only reason for ∂_t est's is to cancel chance to control w/ 's ...

~~PSS: throw in $\hat\Delta\phi$ (...) to flip S~~ $(S)^k_t$

Song-Weinkove: cook it up w/ tr$_{\omega_0}\omega$ to chg. sign (S). Use Hamilton's inductive + allows to bd $\nabla^m_g S$ $\quad$ f'las to bootstrap to $\nabla^m Rm$.

Paper II:

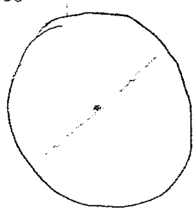
~~T/P deg. ... Read GH ... p. 2 notes~~

Orbifold singularities
here ≠ blow-downs of $\mathbb{P}^{n-1}$'s w/ normal bundle $\mathcal{O}(-k) = L = \{[z_1, \dots, z_n], \sigma\} \subset \mathbb{P}^{n-1} \times \mathbb{C}^n$

$\sigma \propto (z_1^k, \dots, z_n^k)$

Chart $D_i \ni (w^j_{(i)}, y_i) \mapsto ([w^1, \dots,], y_i(w^1, \dots, 1, \dots))$
σ

Same picture ... [illegible]



Define $\pi : L \to \mathbb{C}^n / \mathbb{Z}_k$ $\mathrm{diag}(e^{2\pi i/k})$

$([z], (\lambda z_1^k, \lambda z_2^k, \dots, \lambda z_n^k)) \mapsto \sqrt[k]{\lambda}\,(z_1, \dots, z_n)$. Well-def'd by 2nd coord.

$D_i \ni (w, y) \mapsto \sqrt[k]{y_i}\,(w^1_{(i)}, \dots, w^n_{(i)}) \in \mathbb{C}^n/\mathbb{Z}_k$.

(could do this on Blowup, but not good w/ charts.) (2nd paper: long notat'n and mess.)

But doesn't lift to orbifold chart ⇒ not smooth orb. map.
In fact, $\pi^*(\omega^{\mathbb{C}^n}_{Eucl})$ not smooth on L.
↳ smooth orbifold Kähler metric. ($\mathbb{Z}_k$-invariant.)

Remedy: define a section s of $[E]$,

$$\{s_i|_{D_i} = y_{(i)}\}$$

Metric h on L s.t.

$$|s|_h^2 = |y_{(i)}|^2 \left(\frac{\sum_{j=1}^n |z_j|^2}{|z_i|^2}\right)^k = \pi^* r^{2k}.$$

Also def: $\omega_\infty = \frac{i}{2\pi}\partial\bar\partial(r^{2k})$, nonneg. orbifold (1,1)-form.

$$\pi^*\omega_\infty = \frac{i}{2\pi}\partial\bar\partial|s|_h^2, \text{ smooth}.$$

Lemma $\omega := \pi^*\omega_\infty - c\,R(h)$ is Kähler on L.

And pushfwd is invariant, doesn't ... too badly at O.
Use this to argue that $\exists f$ s.t. $\tilde\omega_X = \pi^*(\omega_{orb} + \frac{i}{2\pi}\partial\bar\partial f)$ smooth on X.
So by changing within $[\omega_{orb}]$, can pull back smoothly $\pi^*\omega_X$, carry over Rey'est's.

BTW: RF preserves orbifold metrics; they show it turns blowdown into smooth Kähler orb. metric for $t > T$. (10)

<u>Surfaces</u>: Endgame studies.

- Paper II does Hb surf contraction of D_0 w/o assuming Calabi symmetry
  $\Longrightarrow$ End-game in Fano case.

- Paper I shows <u>general type</u> gets contracted to smooth min'l model. (Adjunction $\Rightarrow$ exceptional $\mathbb{P}^1$'s are (-1).) that get contracted by RF

Then paper II proves that desired endgame occurs:
on min'l model have some (-2)-curves (not $\mathbb{P}^1$) that make Tsuji metric singular. In fact, canonical model has only orbifold (-2)-sing's $\Longrightarrow$ normalized limit is a smooth <u>orbifold</u> KE metric (Kobayashi.)
better than degen. RF on desing'n.